IB Mathematics Analysis and Approaches HL

Bottle Design Efficiency: A Volume-to-Surface Area Analysis of Pepsi, Coca-Cola, Fuze, and Smartwater

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Table of Contents

1. Introduction

1.1. Rationale

Bottled drinks are widely consumed worldwide, and in 2018, an average person drank 2.7 servings of soft drinks per week (Caputo, 2023). It is estimated that 1 million plastic bottles are used every minute (Pace, 2019). From 1950 to 2015, there was a total of 8.3 billion tons of plastic produced, with 4.9 billion tons, or 59%, discarded (Barrett, 2019). The rate of plastic production is growing fast and global production is predicted to reach 1,100 million tonnes by 2050, with 85% ending up in landfills or as unregulated waste (UNEP, 2022). The plastic bottles that are thrown away are discarded in landfills, littered on the streets, or released into marine environments (Franklin, 2023). Plastic is not biodegradable and can take 1000 years to decompose, meaning it will just continuously accumulate in the environment (UNEP, 2023). The disposal of bottles leads to a cost of 0.7 to 29.2 billion USD a year (Tamburini et al., 2025). Additionally, the production, transportation, and low recycling rates of plastic bottles further harm the environment (Centers, 2015). Therefore, by designing bottles that require less plastic while maintaining the same volume, companies can reduce material use and plastic waste, thereby minimizing environmental impact and lowering production cost.

1.2. Aim

This investigation evaluates the shape efficiency of bottle designs of different drink brands. The most efficient shape maximizes the drink volume while minimizing the surface area. The exploration examines plastic bottles from PepsiCo, Inc. and the

Coca-Cola Company, including Fuze and Glaceau Smartwater as Coca-Cola brands, referred to as Pepsi, Coca-Cola (or Coke), Fuze and Smartwater (or Water), respectively. Efficiency is calculated with a volume-to-surface area ratio of each bottle.

1.3. Background

The four plastic bottles with common sizes of 500 mL and 591 mL of two major beverage brands are chosen to be modelled as shown in Figure 1. They are chosen because of their varied bottle styles and different types of drinks, and also being popular drinks means that many bottles are produced. The first bottle image is Pepsi (Walmart, n.d.-c), the other three bottles include Coca-Cola (Walmart, n.d.-a), Fuze Iced Tea (Saveonfoods, 2025) and Smartwater (Walmart, n.d.-b), respectively.

Figure 1: Bottle Images That Will be Modelled (See above sentence for sources)

The volume and surface area of the bottles can be calculated through mathematical modelling and integration, enabling the evaluation of shape efficiency. This analysis assumes the images are not warped and accurately represent the real bottles. It also supposes that each bottle has a uniform shape all around as the graphing of the model shows only one side.

2. Calculations

2.1. Modelling

The bottles are modelled using the mathematical tool, GeoGebra. The images are horizontally centered on the x-axis, and the positive half is modelled with functions, as shown in Appendix II. The functions are attained by plotting points using the guide of the images, and using GeoGebra's FitPoly function to generate a polynomial that best fits the points. They include polynomials of power 0 to 7, with 0 representing a horizontal line. For clarity, the functions in the calculations are rounded to 2 decimals.

Since the functions are obtained based on the outer edge of the bottles, wall thickness can increase its estimated volume, particularly for the Fuze bottle. The bottle wall has a thickness of 0.035 cm, while the neck section, as given in eq1 of Figure 4, has a thickness of 0.30 cm, which was determined by measuring the actual bottle with a caliper. Thus, the thickness is subtracted from the equations to get a more accurate volume.

Figures 2-5 are the modelled bottles on GeoGebra (Geogebra, 2001). The full images are in Appendix I. 1 unit on the graph corresponds to 1 cm of the actual bottle, with the real bottle height measured with a ruler and the greatest bottle diameter with a caliper.

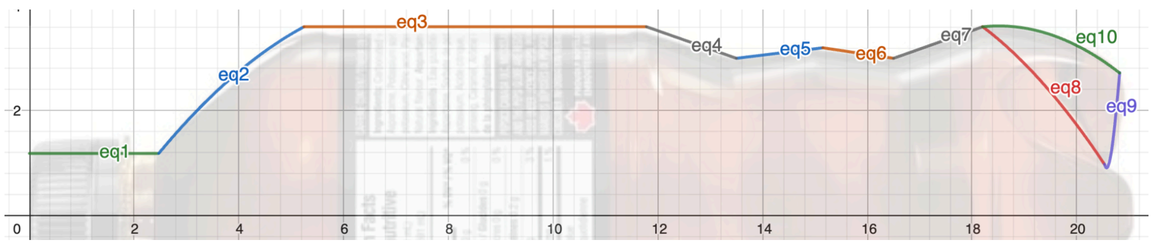

Figure 2: Modelled Pepsi Bottle (Walmart, n.d.-c)

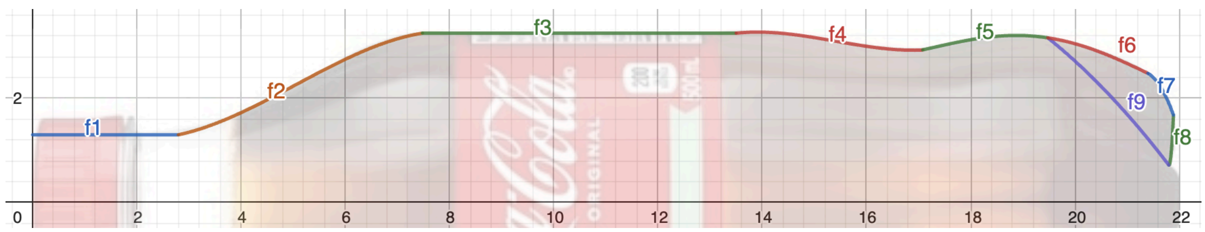

Figure 3: Modelled Coca-Cola Bottle (Walmart, n.d.-a)

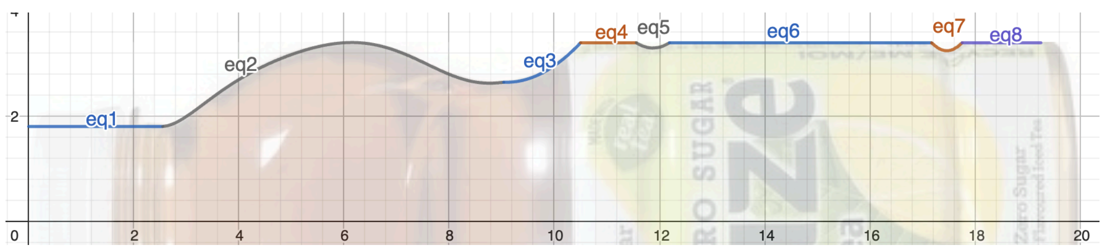

Figure 4: Modelled Fuze Bottle (Saveonfoods, 2025)

Figure 5: Modelled Smartwater Bottle (Walmart, n.d.-b)

2.2. Formula Derivation

I. Variables

Table 1: Variables Used in the Investigation

Variable	Definition
a	x value at the start of the function
b	x value at the end of the function
n	Number of functions to model the bottle
y	Equation of the function
t	Thickness of the bottle
V	Volume
A	Surface area

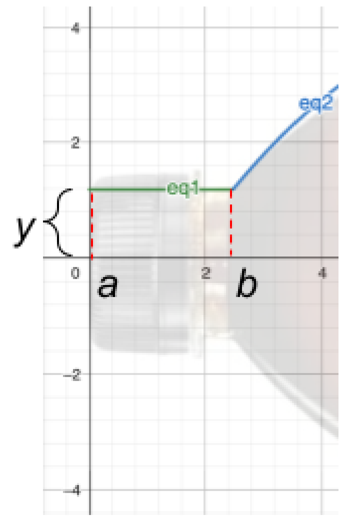

Figure 6: Labelled Bottle Section (Walmart, n.d.-c)

II. Volume

1) Calculating volume for one section of the bottle:

$$A_{circle} = \pi r^2 \quad [1]$$

The radius r from Equation [1] is equivalent to the function $y = f(x)$:

$$V = \int_a^b \pi y^2 dx$$

$$V = \pi \int_a^b [f(x)]^2 dx \quad [2]$$

2) The volume of revolution for the whole bottle:

$$V = \pi \sum_{k=1}^{n} \int_a^b [f_k(x)]^2 dx \quad [3]$$

III. Surface Area

1) Calculating surface area for the whole bottle excluding the base:

The surface area is the sum of the circumferences of infinite circular disks of the section. Each section is modelled to get a function.

$$C = 2\pi r$$

$$A_{circumference} = \int_a^b 2\pi y dx, \; y = f(x)$$

$$A_{circumference} = 2\pi \int_a^b f(x) dx \qquad [4]$$

The following equation is the sum of the surface area of the individual functions of the bottle:

$$A = 2\pi \sum_{k=1}^{n} \left[\int_{a_k}^{b_k} f_k(x) dx\right] \qquad [5]$$

2) Calculating surface area for the base of the bottle:

The base area (A_{base}) is added as part of the surface area, and the cap area is not included. The base radius is r_{base}.

$$A_{base} = \pi (r_{base})^2 \qquad [6]$$

3) Surface Area for the whole bottle is the sum of Equations [5] and [6]:

$$A = 2\pi \sum_{k=1}^{n} \left[\int_{a_k}^{b_k} f_k(x) dx\right] + \pi (r_{base})^2 \qquad [7]$$

2.3. Calculating Volume and Surface Area

Sample calculation with eq4 of Pepsi in Figure 2 (values rounded to 2 decimals):

$$f(x) =- 0.35x + 7.72, \{11.78 \le x < 13.50\}$$

1) Solve for the volume of eq4 using Equation [2]:

$$V = \pi \int_{a}^{b} (f(x))^2 dx$$

$$V = \pi \int_{11.78}^{13.50} (-0.35x + 7.72)^2 dx$$

$$= \pi [\frac{(-0.35x+7.72)^3}{-0.35\times3}]_{11.78}^{13.50}$$

$$= \pi [\frac{(-0.35x+7.72)^3}{-1.05}]_{11.78}^{13.50}$$

Using the fundamental theorem of calculus:

$$= \pi (\frac{(-0.35\times13.50+7.72)^3}{-1.05} - \frac{(-0.35\times11.78+7.72)^3}{-1.05})$$

$$= \pi (\frac{(-4.72+7.72)^3}{-1.05} - \frac{(-4.12+7.72)^3}{-1.05})$$

$$= \pi (\frac{3.00^3}{-1.05} - \frac{3.60^3}{-1.05})$$

$$\approx 18.79\pi \approx 59.03\ cm^3$$

2) Solving for total volume of the Pepsi bottle:

Code is written to perform integration for the rest of the volume calculations, which is given in Appendix III, with the results presented in Appendix II, Table 6. The total volume of the Pepsi bottle is determined with Equation [3], by adding the volumes of the individual functions.

$$V_{total} = 10.99 + 60.76 + 266.28 + ... + 47.75 + 2.37$$

$$= 593.04\ cm^3$$

3) Calculating the surface area of Pepsi eq4 in Figure 2 with Equation [4]:

$$A_{circumference} = 2\pi \int_a^b f(x)dx$$

$$= 2\pi \int_{11.78}^{13.5} (-\ 0.35x + 7.72)\ dx$$

$$= 2\pi[-\ 0.17x^2 + 7.72x]_{11.78}^{13.50}$$

$$= 2\pi \times [-\ 0.17(13.50)^2 + 7.72(13.50) - (-\ 0.17(11.78)^2 + 7.72(11.78)]$$

$$= 11.35\pi \approx 35.67\ cm^2$$

Calculate the rest surface areas using the code in Appendix III, the results are given in Appendix II, Table 6.

4) Sample calculation of the Pepsi base surface area with Equation [6]:

$$A_{base} = \pi(r_{base})^2$$

The function that ends at the base is eq10 with equation (rounded to 2 decimals):

$$f(x) =-\ 52.94x^3 + 3309.44x^2 - 68946.38\,x + 478717.54,\ \{20.53 \le x < 20.82\}$$

x is $20.82\ cm$, as that is the end value. Substituting in x gives:

$$A_{base} = \pi(2.75)^2 \approx 23.84\ cm^2$$

5) Calculate the total surface area of the Pepsi bottle

Using Equation [7], which is the sum of the function surface areas and the base:

$$A_{total} = 18.47 + 44.36 + 147.93 + ... + 35.70 + 2.77 = 370.85\ cm^2$$

The rest of the data is added up using Google Sheets and given in Appendix II. The volume for Pepsi and Coca-Cola are taken with the average of the elevated and lowered base functions.

2.4. Comparing Volume-to-Surface Area Ratios

The volume-to-surface area ratio is calculated with the following equation:

$$V: A\ Ratio = \frac{V}{A}$$

Sample Calculation with Pepsi Bottle:

$$V: A\ Ratio = \frac{593.04}{417.73} = 1.42\ cm$$

The volume-to-surface area ratios for all the bottles are given in Table 2. The base surface area for Pepsi and Coca-Cola are taken with the function that ends at the base.

Table 2: Total Volume, Surface Area and Ratio Summary of the Plastic Bottles

Bottle Types	**Modelled Volume** (cm^3)	**Volume Difference** (cm^3)	**Surface Area** (cm^2)	***V*:*A* Ratio** (cm)
Pepsi (591 mL)	593.04	2.04	417.73	1.42
Water (591 mL)	598.11	7.11	426.46	1.40
Coke (500 mL)	503.19	3.19	384.30	1.31
Fuze (500 mL)	526.69	26.69	391.62	1.34

As shown in Table 2, the modelled volume closely matches the measured volume for each bottle, except for the 500 mL Fuze bottle, which is about 26.69 mL larger. The main reason is because the Fuze bottle has a complex base that curves inwards, while the calculation is based on a flat base.

The Pepsi and Water bottles, both with a volume of 591 mL, have larger volume-to-surface area ratios than the Coke and Fuze bottles, which have a volume of 500 mL. This indicates that larger volume bottles are more efficient in design. Among the four bottle types, Pepsi has the highest design efficiency. Although the Fuze bottle ratio is higher than the same volume Coke bottle, the assumption of a flat base instead of a punt base in the calculation results in a slightly larger volume and lower surface area, resulting in a higher volume-to-surface area ratio.

2.5. Material Efficiency

The thickness of the plastic bottle body affects the amount of material used and the effective volume of the bottles. The bottle wall is too thin to be measured directly with a caliper, thus, 10 pieces are cut from each bottle and put tightly together to measure the wall thickness. The results are given in Table 3. Both Pepsi and Coke have smaller thicknesses, while Fuze has the greatest. The bottle body includes the base but excludes the neck. It is assumed that the thickness of the body is constant throughout. The total material used for a bottle is calculated using Equation [8]:

$$V_{Material} = A_{neck} \times t_{neck} + A_{body} \times t_{body} \qquad [8]$$

Sample Calculation for Pepsi Bottle Material Volume using Equation [8]:

$$V_{Material} = 18.47 \times 0.20 + 399.26 \times 0.023 = 12.88\, cm^3$$

The Material Efficiency Ratio (MER) is the modelled total bottle volume divided by the total plastic volume. Sample calculation for the material efficiency ratio:

$$MER = \frac{V_{total}}{V_{material}}$$

$$= \frac{593.04}{12.88} = 46.05$$

Table 3: Material Efficiency of the Chosen Bottles

Bottle Types	Neck *A* (cm^2)	Body *A* (cm^2)	Neck Thickness (cm)	Body Thickness (cm)	Material Volume (cm^3)	Material Efficiency Ratio
Pepsi (591 mL)	18.47	399.26	0.20	0.023	12.88	46.05
Water (591 mL)	28.85	397.61	0.20	0.033	18.89	31.66
Coke (500 mL)	22.87	361.43	0.20	0.024	13.25	37.98
Fuze (500 mL)	29.00	362.62	0.30	0.035	21.39	24.62

For the 591 mL bottles, Pepsi has the highest MER, which means that it uses less plastic to hold the same amount of beverage as Smartwater. Coke is also more efficient than the other 500 mL bottle, Fuze. These more efficient designs allow them to pay less for material, and create less plastic pollution.

3. Optimized Design

To optimize the design of the bottle, the shape is simplified into a cone and a cylinder. The surface areas are calculated with the sides of the cone excluding the base, and the sides and base of the cylinder, excluding the top. This calculates only the outer surface area of the bottle, which is an accurate representation.

3.1. Cone

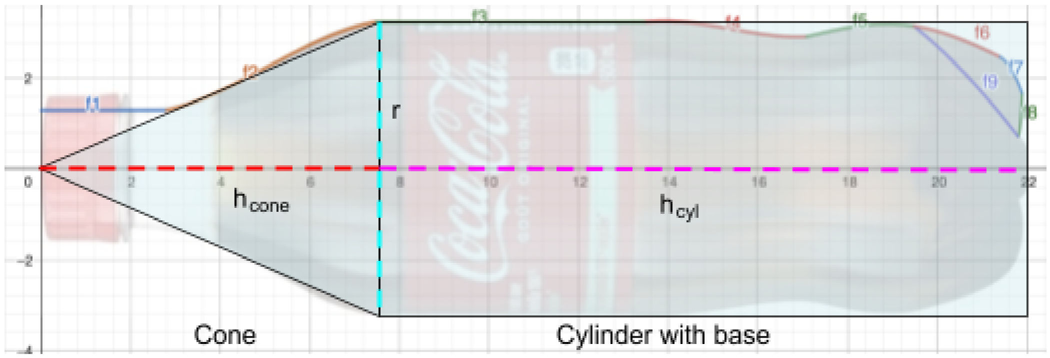

Figure 8: Simplified Bottle Shape (Walmart, n.d.-c)

The formula for the volume of a cone is Equation [9]:

$$V_{cone} = \frac{1}{3}\pi r^2 h_{cone} \qquad [9]$$

$$h_{cone} = \frac{3V_{cone}}{\pi r^2} \qquad [10]$$

The following area does not include the base of the cone:

$$A_{cone} = \pi r\sqrt{r^2 + h_{cone}^2}$$

Replace h_{cone} with Equation [10]:

$$A_{cone} = \pi r\sqrt{r^2 + (\frac{3V_{cone}}{\pi r^2})^2}$$

$$A_{cone} = \frac{\sqrt{\pi^2 r^6 + 9V_{cone}^2}}{r}$$

$$\frac{dA_{cone}}{dr} = \frac{-\sqrt{\pi^2 r^6 + 9V_{cone}^2} + \frac{3\pi^2 r^6}{\sqrt{\pi^2 r^6 + 9V_{cone}^2}}}{r^2}$$

$$= \frac{2\pi^2 r^6 - 9V_{cone}^2}{r^2\sqrt{\pi^2 r^6 + 9V_{cone}^2}} \quad [11]$$

$\frac{dA}{dr}$ needs to equal to 0 to find the turning point:

$$\frac{dA_{cone}}{dr} = \frac{3\pi^2 r^4}{\sqrt{\pi^2 r^6 + 9V_{cone}^2}} - \frac{\sqrt{\pi^2 r^6 + 9V_{cone}^2}}{r^2} = 0$$

$$3\pi^2 r^4 \times r^2 = \pi^2 r^6 + 9V_{cone}^2$$

$$V_{cone} = \frac{\sqrt{2}}{3}\pi r^3$$

Finding the second derivative from Equation [11]:

$$\frac{d^2 A_{cone}}{dr^2} = \frac{\frac{d}{dr}\left[2\pi^2 r^6 - 9V_{cone}^2\right] \times r^2\sqrt{\pi^2 r^6 + 9V_{cone}^2} - \left(2\pi^2 r^6 - 9V_{cone}^2\right) \times \frac{d}{dr}\left[r^2\sqrt{\pi^2 r^6 + 9V_{cone}^2}\right]}{\left(r^2\sqrt{\pi^2 r^6 + 9V_{cone}^2}\right)^2}$$

$$= \frac{(2\pi^2 \times \frac{d}{dr}\left[r^6\right] + \frac{d}{dr}\left[-9V_{cone}^2\right] r^2\sqrt{\pi^2 r^6 + 9V_{cone}^2}}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$- \frac{\left(2\pi^2 r^6 - 9V_{cone}^2\right)\left(\frac{d}{dr}\left[r^2\right] \times \sqrt{\pi^2 r^6 + 9V_{cone}^2} + r^2 \times \frac{d}{dr}\left[\sqrt{\pi^2 r^6 + 9V_{cone}^2}\right]\right)}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$= \frac{\left(2\pi^2 \times 6r^5 + 0\right) r^2\sqrt{\pi^2 r^6 + 9V_{cone}^2}}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$- \frac{\left(2\pi^2 r^6 - 9V_{cone}^2\right)\left(2r\sqrt{\pi^2 r^6 + 9V_{cone}^2} + r^2 \times \frac{1}{2\sqrt{\pi^2 r^6 + 9V_{cone}^2}} \times \frac{d}{dr}\left[\pi^2 r^6 + 9V_{cone}^2\right]\right)}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$= \frac{12\pi^2 r^7 \sqrt{\pi^2 r^6 + 9V_{cone}^2} - \left(2\pi^2 r^6 - 9V_{cone}^2\right)\left(2r\sqrt{\pi^2 r^6 + 9V_{cone}^2} + \frac{r^2(\pi^2 \times \frac{d}{dr}[r^6] + \frac{d}{dr}[9V_{cone}^2])}{2\sqrt{\pi^2 r^6 + 9V_{cone}^2}}\right)}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$= \frac{12\pi^2 r^7 \sqrt{\pi^2 r^6 + 9V_{cone}^2} - \left(2\pi^2 r^6 - 9V_{cone}^2\right)\left(2r\sqrt{\pi^2 r^6 + 9V_{cone}^2} + \frac{r^2(\pi^2 \times 6r^5 + 0)}{2\sqrt{\pi^2 r^6 + 9V_{cone}^2}}\right)}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$= \frac{12\pi^2 r^7 \sqrt{\pi^2 r^6 + 9V_{cone}^2} - \left(2\pi^2 r^6 - 9V_{cone}^2\right)\left(2r\sqrt{\pi^2 r^6 + 9V_{cone}^2} + \frac{3\pi^2 r^7)}{\sqrt{\pi^2 r^6 + 9V_{cone}^2}}\right)}{r^4\left(\pi^2 r^6 + 9V_{cone}^2\right)}$$

$$= \frac{12\pi^2 r^3}{\sqrt{\pi^2 r^6 + 9V_{cone}^2}} - \frac{2(2\pi^2 r^6 - 9V_{cone}^2)}{r^3\sqrt{\pi^2 r^6 + 9V_{cone}^2}} - \frac{3\pi^2 r^3(2\pi^2 r^6 + 9V_{cone}^2)}{\sqrt{(\pi^2 r^6 + 9V_{cone}^2)^3}}$$

$$= \frac{2\pi^4 r^{12} + 117\pi^2 V_{cone}^2 r^6 + 162V_{cone}^4}{r^3\sqrt{(\pi^2 r^6 + 9V_{cone}^2)^3}} > 0$$

Therefore, given a specific volume, the surface area is a minimum, which maximizes the efficiency.

With Equation [9], $V_{cone} = \frac{1}{3}\pi r^2 h_{cone} = \frac{\sqrt{2}}{3}\pi r^3$

$$h_{cone} = \sqrt{2}r \quad [12]$$

$$A_{cone} = \pi r\sqrt{r^2 + h_{cone}^2} = \sqrt{3}\pi r^2$$

3.2. Cylinder

The formula for the volume of a cylinder is as follows, it does not include the top of the cylinder, but includes the base:

$$V_{cyl} = \pi r^2 h_{cyl}$$

$$h_{cyl} = \frac{V_{cyl}}{\pi r^2}$$

$$A_{cyl} = 2\pi r h_{cyl} + \pi r^2$$

3.3. Total: Cone + Cylinder

The total optimized bottle volume is the sum of the volumes of the cone and cylinder. Both the cone and cylinder have the same radius.

$$V_{total} = V_{cone} + V_{cyl}$$

$$V_{total} = \frac{\sqrt{2}}{3}\pi r^3 + \pi r^2 h_{cyl} \quad [13]$$

$$h_{cyl} = \frac{V_{total} - \frac{\sqrt{2}}{3}\pi r^3}{\pi r^2}$$

The total optimized bottle surface area is the sum of the cylinder and cone surface areas.

$$A_{total} = A_{cone} + A_{cyl}$$

$$A_{total} = \sqrt{3}\pi r^2 + 2\pi r h_{cyl} + \pi r^2$$

$$A_{total} = \sqrt{3}\pi r^2 + 2\pi r\left(\frac{V_{total} - \frac{\sqrt{2}}{3}\pi r^3}{\pi r^2}\right) + \pi r^2$$

$$A_{total} = \frac{2V_{total}}{r} + \left(1 + \sqrt{3} - \frac{2\sqrt{2}}{3}\right)\pi r^2 \quad [14]$$

Differentiate A_{total} to r: $\frac{dA_{total}}{dr} = -\frac{2V_{total}}{r^2} + 2\left(1 + \sqrt{3} - \frac{2\sqrt{2}}{3}\right)\pi r$

$\frac{dA}{dr}$ is equal to 0 to find the turning point:

$$\frac{dA_{total}}{dr} = -\frac{2V_{total}}{r^2} + 2\left(1 + \sqrt{3} - \frac{2\sqrt{2}}{3}\right)\pi r = 0$$

$$V_{total} = (1 + \sqrt{3} - \frac{2\sqrt{2}}{3})\pi r^{3} \quad [15]$$

Finding the second derivative to find the maximum or minimum:

$$\frac{d^{2}A_{total}}{dr^{2}} = \frac{4V_{total}}{r^{3}} + 2(1 + \sqrt{3} - \frac{2\sqrt{2}}{3})\pi r^{2} > 0$$

Therefore, the surface area is a minimum, which maximizes the efficiency.

Equation [15] is equal to Equation [13]:

$$V_{total} = (1 + \sqrt{3} - \frac{2\sqrt{2}}{3})\pi r^{3} = \pi r^{2}h_{cyl} + \frac{\sqrt{2}}{3}\pi r^{3}$$

$$h_{cyl} = (1 + \sqrt{3} - \sqrt{2})r \approx 1.32r \quad [16]$$

3.4. Optimizing Efficiency for a 500 mL bottle

Using Equation [15]:

$$V_{total} = (1 + \sqrt{3} - \frac{2\sqrt{2}}{3})\pi r^{3} = 500\ cm^{3}$$

$$5.62r^{3} = 500$$

$$r = 4.46\ cm \quad [17]$$

From Equation [16] and Equation [12]:

$$h_{total} = h_{cone} + h_{cyl}$$

$$h_{total} = \sqrt{2}r + 1.32r$$

Substituting in r from Equation [17]:

$$h_{total} = 12.19\ cm$$

Solving for Surface Area with Equation [14]:

$$A_{total} = \frac{2V_{total}}{r} + (1 + \sqrt{3} - \frac{2\sqrt{2}}{3})\pi r^{2}$$

$$= \frac{2\times500}{4.46} + 5.62\times\ 4.46^2$$

$$= 336.01\ cm^2$$

Solving for $V : A$ Ratio:

$$\frac{V_{total}}{A_{total}} = \frac{500}{336.01} = 1.49\ cm$$

Table 4: Optimized Volume to Area Ratio

Bottle Types	***r*** (cm)	***h*** (cm)	***A*** (cm^2)	**Optimized *V*:*A* Ratio** (cm)
500 mL	4.46	12.19	336.01	1.49
591 mL	4.72	12.90	375.63	1.57

For a 500 mL bottle, the optimized bottle has a volume-to-surface area ratio of 1.49 cm, with a radius of 4.46 cm and a height of 12.19 cm. While a 591 mL bottle has a volume-to-surface area ratio of 1.57 cm, radius of 4.72 cm, and height of 12.90 cm. Both have a higher efficiency by 10 to 14% compared to the four selected bottles.

4. Conclusion

According to the calculations, the 591 mL Pepsi bottle has the highest volume-to-surface area ratio, while the 500 mL Fuze bottle has the lowest ratio. In considering the thickness of the bottles, Pepsi and Coca-Cola use less plastic to hold the same volume of beverage as Smartwater and Fuze. The optimization was found by simplifying the bottle to a cone and a cylinder. Even the 591 mL Pepsi bottle design can still be improved. Based on the optimization, the efficiency of the two sizes of bottles can be increased by up to 14%. However, the optimization is only based on the volume-to-surface area efficiency. A product designed for the real-world must also consider factors such as ergonomic fit in the hand, grip to prevent slipping and neck shape for easy drinking.

5. Evaluation

5.1. Strengths

I. Same method for all bottles

Using the same graphing method for all the bottles creates consistent results that can be compared. It ensures that there are no discrepancies due to the methodology. It allows for objectivity across the different bottles.

II. Multiple functions and points allowed it to fit well

Modelling using multiple functions and points on the bottle fits the model to the shape better resulting in a more accurate calculation of the volume and surface area.

III. Optimization is general to all the bottles

The bottle optimization uses a general form that consists of a cone and a cylinder, making it applicable to all four bottles. Though it is simplified, the method can effectively represent the selected bottles.

5.2. Limitations

I. Bottle image distorted

Photographs of the bottles may have slight angle distortions at both ends, causing difficulty in defining the endpoints. Variations in image angles can introduce inconsistencies. To minimize this effect, images were selected to have the flattest possible ends, but they are not completely perfect. A possible solution is to capture smaller parts of the bottle separately to reduce the angle distortion.

II. Bottle image position symmetrical on the coordinate plane

When positioning the bottle image on the coordinate plane, it is important that the x-axis passes through the exact center of the bottle to ensure symmetry. Since only the top half is modelled, the upper and lower halves must be identical. To minimize incorrect alignment, the images are cropped as closely as possible to the bottle's edges. However, it would be difficult to achieve perfect centering.

III. Model is not exact

The modelled curves may not perfectly match the actual bottle shape, which introduces error. Additionally, it is modelled on the outer perimeter rather than the inside, leading to an overestimation of volume. Variations in wall thickness among the bottles can further contribute to error. The bottle bases are simplified to be flat, instead of capturing their complex inward curvature. All of these collectively result in the modelled volumes of the bottles being greater than the actual volume, especially for the Fuze bottle.

IV. Patterns on the Bottles

The Pepsi, Coca-Cola and Fuze bottles have small ridges and patterns that the model does not follow exactly. Since these features curve inwards, the modelled volume is slightly overestimated compared to the actual volume. Additionally, the irregularly shaped bases of the bottles makes modeling more difficult.

6. Bibliography

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7. Appendix I: Modelled Bottles

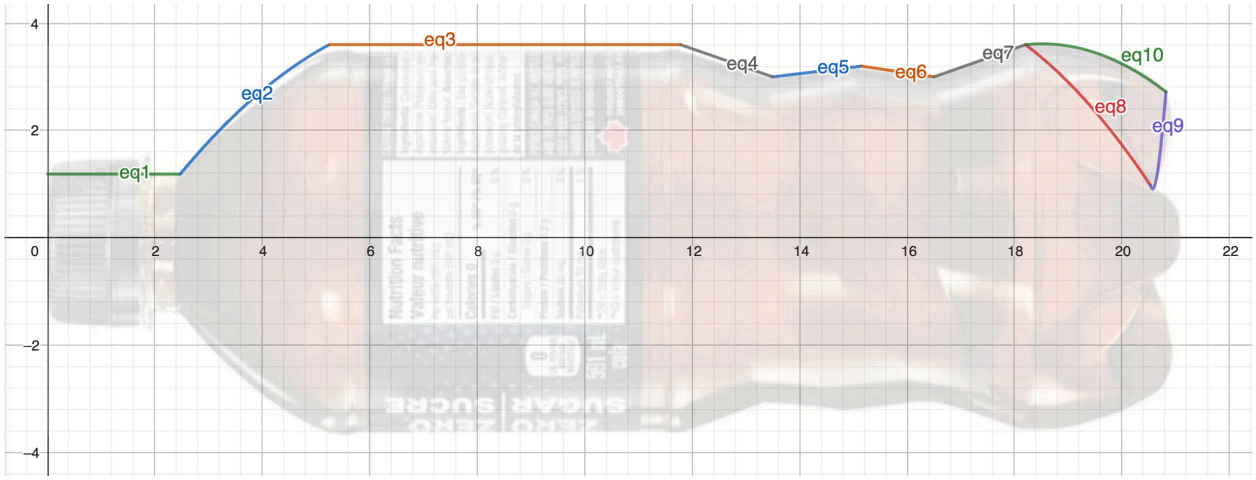

Figure 7: Modelled Pepsi Bottle (Walmart, n.d.-c) (eq10 and eq9 are the elevated part of the base and eq8 is the lowered portion)

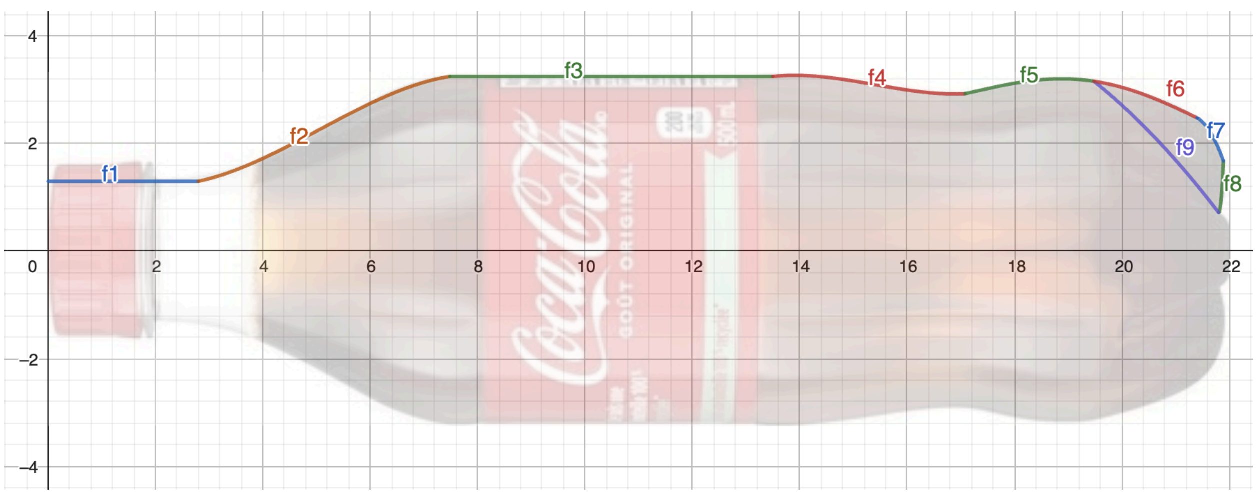

Figure 8: Modelled Coca-Cola Bottle (Walmart, n.d.-a) (f6, f7 and f8 are the elevated part of the base and f9 is the lowered portion)

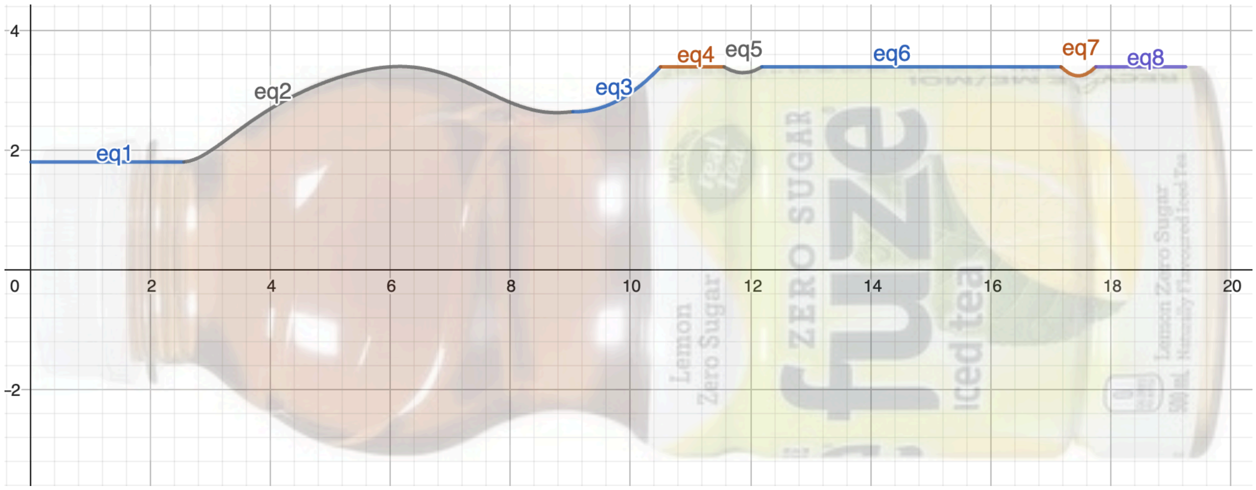

Figure 9: Modelled Fuze Bottle (Saveonfoods, 2025)

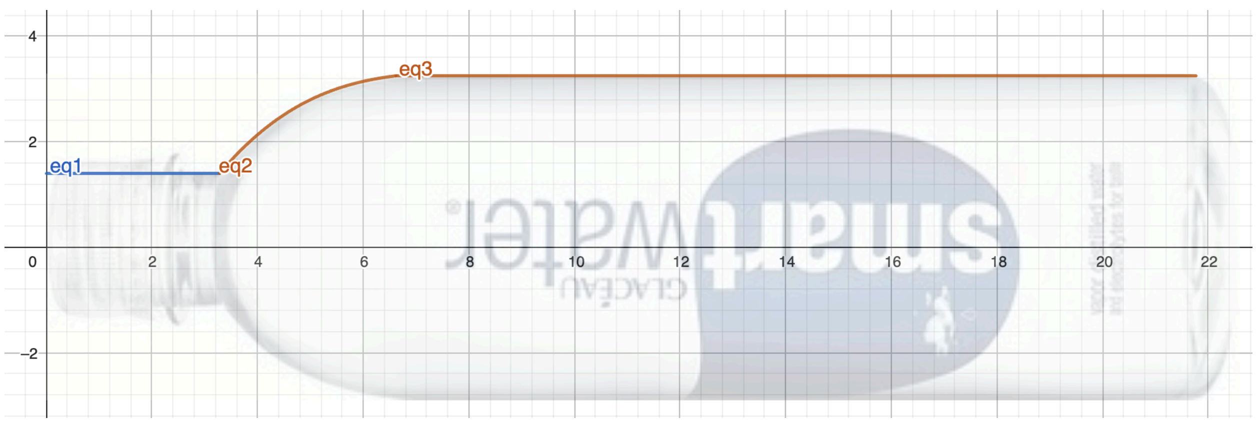

Figure 10: Modelled Smartwater Bottle (Walmart, n.d.-b)

8. Appendix II: Tables for Functions, Volume and Surface Area

Table 5: Functions to Model the Pepsi Bottle

Label	Function (cm)	Range (cm)
eq1	$\frac{119}{100}$	$\{0 \leq x < 2.47\}$
eq2	$-0.12034x^2 + 1.79685x - 2.51299$	$\{2.47 \leq x < 5.24\}$
eq3	$\frac{18}{5}$	$\{5.24 \leq x < 11.78\}$
eq4	$-0.34968x + 7.72064$	$\{11.78 \leq x < 13.5\}$
eq5	$0.12121x + 1.36364$	$\{13.5 \leq x < 15.15\}$
eq6	$-0.14799x + 5.44205$	$\{15.15 \leq x < 16.5\}$
eq7	$0.35231x - 2.81361$	$\{16.5 \leq x < 18.2\}$
eq8*	$-0.1681x^2 + 6.22481x - 53.9977$	$\{18.22 \leq x < 20.82\}$
eq9*	$-0.16102x^2 + 5.11093x - 36.07891$	$\{18.22 \leq x < 20.53\}$
eq10*	$-52.9425x^3 + 3309.4357x^2 - 68946.3849x + 478717.53752$	$\{20.53 \leq x \leq 20.82\}$

*eq10 and eq9 are the elevated part of the base and eq8 is the lowered portion.

Table 6: Volume and Surface Area of Pepsi Bottle

Label	Volume (cm^3)	Surface Area (cm^2)	Base Area (cm^2)
eq1	10.99	18.47	23.84
eq2	60.76	44.36	
eq3	266.28	147.93	
eq4	59.03	35.67	
eq5	49.83	32.14	
eq6	40.77	26.30	
eq7	58.28	35.24	
eq8	91.14	53.79	
eq9	47.75	35.70	
eq10	2.37	2.77	
Total	593.04	370.85	

Table 7: Functions to Model the Coca-Cola Bottle

Label	Function (cm)	Range (cm)
f1	$\frac{13}{10}$	$\{0.0 \leq x < 2.8\}$
f2	$-0.02122x^3 + 0.3213x^2 - 1.08845x + 2.29441$	$\{2.8 \leq x < 7.48\}$
f3	$\frac{13}{4}$	$\{7.48 \leq x < 13.5\}$
f4	$0.02607x^3 - 1.20454x^2 + 18.37725x - 89.45885$	$\{13.5 \leq x < 17.08\}$
f5	$0.00499x^4 - 0.38665 + 11.11529x^2 - 140.47804x + 661.85099$	$\{17.08 \leq x < 19.48\}$
f6*	$0.00499x^4 - 0.38665x^3 + 11.11529x^2 - 140.47804x + 661.85099$	$\{19.48 \leq x < 21.4\}$
f7*	$-2.37065x^2 + 100.99258x - 1073.1005$	$\{21.4 \leq x < 21.89\}$
f8*	$133.33333x^2 - 5814.66667x + 63395.12$	$\{21.8 \leq x < 21.89\}$
f9*	$-0.10899x^2 + 3.4473x - 22.63572$	$\{19.48 \leq x \leq 21.8\}$

*f6, f7 and f8 are the elevated part of the base and f9 is the lowered portion.

Table 8: Volume and Surface Area of Coca-Cola Bottle

Label	Volume (cm^3)	Surface Area (cm^2)	Base Area (cm^2)
f1	14.87	22.87	19.32
f2	83.28	67.51	
f3	199.76	122.93	
f4	108.72	69.88	
f5	73.65	47.11	
f6	50.09	34.68	
f7	7.35	6.69	
f8	0.32	0.58	
f9	33.87	29.69	
Total	503.19	348.21	

Table 9: Functions to Model the Fuze Bottle

Label	Function (cm)	Range (cm)
eq1	$\frac{87}{50} - 0.3$	$\{0.0 \leq x < 2.75\}$
eq2	$-0.00026x^7 + 0.01130x^6 - 0.20151x^5 + 1.93548x^4 - 10.81497 + 35.02438x^2 - 59.86860x + 42.85104 - 0.0351$	$\{2.75 \leq x < 9.04\}$
eq3	$0.37571x^2 - 6.82764\,x + 33.66814 - 0.0351$	$\{9.038 \leq x < 10.5\}$
eq4	$\frac{17}{5} - 0.0351$	$\{10.5 \leq x < 11.56\}$
eq5	$1.00806x^2 - 23.94153\,x + 145.45282 - 0.0351$	$\{11.56 \leq x < 12.19\}$
eq6	$\frac{17}{5} - 0.0351$	$\{12.19 \leq x < 17.18\}$
eq7	$1.80288x^2 - 62.99279\,x + 553.49038 - 0.0351$	$\{17.18 \leq x < 17.76\}$
eq8	$\frac{17}{5} - 0.0351$	$\{17.76 \leq x \leq 19.14\}$

Table 10: Volume and Surface Area of Fuze Bottle

Label	Volume (cm^3)	Surface Area (cm^2)	Base Area (cm^2)
eq1	17.91	29.00	36.32
eq2	165.87	113.25	
eq3	37.70	26.56	
eq4	37.71	22.64	
eq5	21.52	13.19	
eq6	177.50	106.60	
eq7	19.40	12.02	
eq8	49.09	32.04	
Total	526.69	391.62	

Table 11: Functions to Model the Smartwater Bottle

Label	Function (cm)	Range (cm)
eq1	$\frac{7}{5}$	$\{0.0 \leq x < 3.28\}$
eq2	$0.01827x^3 - 0.43190x^2 + 3.43572x - 5.86733$	$\{3.28 \leq x < 6.62\}$
eq3	$\frac{13}{4}$	$\{6.62 \leq x \leq 21.77\}$

Table 12: Volume and Surface Area of Smartwater Bottle

Label	Volume (cm^3)	Surface Area (cm^2)	Base Area (cm^2)
eq1	20.20	28.85	33.18
eq2	75.19	55.06	
eq3	502.72	309.37	
Total	598.11	426.46	

9. Appendix III: Code for Solving Integrals

The following is written in the coding language Python using Visual Studio Code.

```
Python
import sympy as sp

def solveDefIntegral(expression, lowerLim, upperLim):
    try:
        # add multiplication sign
        expression = expression.replace(' x', '*x')

        # expression and variable
        expr = sp.sympify(expression)
        x = sp.symbols('x')  # variable always x

        # calculate definite integral
        defIntegral = sp.integrate(expr, (x, lowerLim, upperLim))

        return defIntegral
    except Exception as e:
        return f"Error: {str(e)}"


# functions
print("Functions - one per line, press Enter twice to finish:") # eg, x**2, 2*x
functions = []
while True:
    line = input()
    if line.strip() == "":
        break
    functions.append(line.strip())

# lower limits
print("Lower limits - one per line, press Enter twice to finish:") # eg, 0, 1
lowerLim = []
while True:
    line = input()
    if line.strip() == "":
        break
    lowerLim.append(float(line.strip()))

# upper limits
print("Upper limits - one per line, press Enter twice to finish:") # eg, 4, 6
upperLim = []
```

```
while True:
    line = input()
    if line.strip() == "":
        break
    upperLim.append(float(line.strip()))

# solve definite integral
results = []
for i in range(len(functions)):
    expr = functions[i]
    lowerLim = lowerLim[i]
    upperLim = upperLim[i]

    result = solveDefIntegral(expr, lowerLim, upperLim)
    results.append(result)

# print results
for i in results:
    print(i)
```